

RMO–1990

1. Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same colour at least two of them will always be of the same size (radius). Prove that there are at least 3 balls which lie in the same box have the same colour and have the same size (radius).
2. For all positive real numbers a, b, c prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

3. A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD . Prove that the crease will divide BC in the ratio 5 : 3.
4. Find the remainder when 2^{1990} is divided by 1990.
5. P is any point inside a triangle ABC . The perimeter of the triangle $AB + BC + CA = 2s$. Prove that

$$s < AP + BP + CP < 2s.$$

6. N is a 50 digit number (in the decimal scale). All digits except the 26th digit (from the left) are 1. If N is divisible by 13, find the 26th digit.
7. A censusman on duty visited a house which the lady inmates declined to reveal their individual ages, but said — “we do not mind giving you the sum of the ages of any two ladies you may choose”. Thereupon the censusman said — “In that case please give me the sum of the ages of every possible pair of you”. The gave the sums as follows : 30, 33, 41, 58, 66, 69. The censusman took these figures and happily went away. How did he calculate the individual ages of the ladies from these figures.
8. If the circumcenter and centroid of a triangle coincide, prove that the triangle must be equilateral.

RMO–1991

1. Let P be an interior point of a triangle ABC and AP , BP , CP meet the sides BC , CA , AB in D , E , F respectively. Show that

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

2. If a , b , c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

3. A four-digit number has the following properties :

- (a) it is a perfect square,
- (b) its first two digits are equal to each other,
- (c) its last two digits are equal to each other.

Find all such four-digit numbers.

4. There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

- (a) remove an equal number of balls simultaneously from the urns, and
- (b) double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty.

5. Take any point P_1 on the side BC of a triangle ABC and draw the following chain of lines : P_1P_2 parallel to AC (P_2 on AB), P_2P_3 parallel to BC , P_3P_4 parallel to AB , P_4P_5 parallel to CA , and P_5P_6 parallel to BC . Here P_2 , P_5 lie on AB ; P_3 , P_6 lie on CA ; and P_4 on BC . Show that P_6P_1 is parallel to AB .
6. Find all integer values of a such that the quadratic expression

$$(x + a)(x + 1991) + 1$$

can be factored as a product $(x + b)(x + c)$ where b and c are integers.

7. Prove that $n^4 + 4^n$ is composite for all integer values of n greater than 1.
8. The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

RMO–1992

1. Determine the set of integers n for which $n^2 + 19n + 92$ is a square of an integer.
2. If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, where a, b, c are positive integers with no common factor, prove that $(a + b)$ is the square of an integer.
3. Determine the largest 3-digit prime factor of the integer ${}^{2000}C_{1000}$.
4. $ABCD$ is a cyclic quadrilateral with $AC \perp BD$; AC meets BD at E . Prove that

$$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2,$$

where R is the radius of the circumscribing circle.

5. $ABCD$ is a cyclic quadrilateral; x, y, z are the distances of A from the lines BD, BC, CD respectively. Prove that

$$\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}.$$

6. $ABCD$ is a quadrilateral and P, Q are mid-points of CD, AB respectively. Let AP, DQ meet at X , and BP, CQ meet at Y . Prove that

$$\text{area of } ADX + \text{area of } BCY = \text{area of quadrilateral } PXQY.$$

7. Prove that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \dots + \frac{1}{3001} < 1\frac{1}{3}.$$

8. Solve the system

$$\begin{aligned}(x + y)(x + y + z) &= 18 \\ (y + z)(x + y + z) &= 30 \\ (z + x)(x + y + z) &= 2A\end{aligned}$$

in terms of the parameter A .

9. The cyclic octagon $ABCDEFGH$ has sides a, a, a, a, b, b, b, b respectively. Find the radius of the circle that circumscribes $ABCDEFGH$ in terms of a and b .

RMO–1993

1. Let ABC be an acute-angled triangle and CD be the altitude through C . If $AB = 8$ and $CD = 6$, find the distance between the mid-points of AD and BC .
2. Prove that the ten's digit of any power of 3 is even. [e.g. the ten's digit of $3^6 = 729$ is 2].
3. Suppose $A_1A_2 \dots A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?
4. Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD ; and similarly r_2 is the radius of the circle passing through B and C and touching AD . Show that

$$r_1 + r_2 \geq \frac{5}{8}(a + b).$$

5. Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162.
6. If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that

$$(1 + a)(1 + b)(1 + c)(1 + d) \geq 16.$$

7. In a group of ten persons, each person is asked to write the sum of the ages of all the other 9 persons. If all the ten sums form the 9-element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$ find the individual ages of the persons (assuming them to be whole numbers of years).
8. I have 6 friends and during a vacation I met them during several dinners. I found that I dined with all the 6 exactly on 1 day; with every 5 of them on 2 days; with every 4 of them on 3 days; with every 3 of them on 4 days; with every 2 of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

RMO–1994

1. A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf.
2. In the triangle ABC , the incircle touches the sides BC , CA and AB respectively at D , E and F . If the radius of the incircle is 4 units and if BD , CE and AF are consecutive integers, find the sides of the triangle ABC .
3. Find all 6-digit natural numbers $a_1a_2a_3a_4a_5a_6$ formed by using the digits 1, 2, 3, 4, 5, 6, once each such that the number $a_1a_2 \dots a_k$ is divisible by k , for $1 \leq k \leq 6$.
4. Solve the system of equations for real x and y :

$$\begin{aligned} 5x \left(1 + \frac{1}{x^2 + y^2} \right) &= 12 \\ 5y \left(1 - \frac{1}{x^2 + y^2} \right) &= 4. \end{aligned}$$

5. Let A be a set of 16 positive integers with the property that the product of any two distinct numbers of A will not exceed 1994. Show that there are two numbers a and b in A which are not relatively prime.
6. Let AC and BD be two chords of a circle with center O such that they intersect at right angles inside the circle at the point M . Suppose K and L are the mid-points of the chord AB and CD respectively. Prove that $OKML$ is a parallelogram.
7. Find the number of all rational numbers m/n such that
 - (a) $0 < m/n < 1$
 - (b) m and n are relatively prime
 - (c) $mn = 25!$
8. If a , b and c are positive real numbers such that $a + b + c = 1$, prove that

$$(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c).$$

RMO–1996

1. The sides of a triangle are three consecutive integers and its inradius is four units. Determine the circumradius.
2. Find all triples (a, b, c) of positive integers such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 3.$$

3. Solve for real number x and y :

$$\begin{aligned} xy^2 &= 15x^2 + 17xy + 15y^2 \\ x^2y &= 20x^2 + 3y^2. \end{aligned}$$

4. Suppose N is an n -digit positive integer such that

- (a) all the n -digits are distinct; and
- (b) the sum of any three consecutive digits is divisible by 5.

Prove that n is at most 6. Further, show that starting with any digit one can find a six-digit number with these properties.

5. Let ABC be a triangle and h_a the altitude through A . Prove that

$$(b + c)^2 \geq a^2 + 4h_a^2.$$

(As usual a, b, c denote the sides BC, CA, AB respectively.)

6. Given any positive integer n show that there are two positive rational numbers a and b , $a \neq b$, which are not integers and which are such that $a - b, a^2 - b^2, a^3 - b^3, \dots, a^n - b^n$ are all integers.
7. If A is a fifty-element subset of the set $\{1, 2, 3, \dots, 100\}$ such that no two numbers from A add up to 100 show that A contains a square.

RMO–1997

1. Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 13$, find $[AFPE]$. (Here $[\cdot]$ denotes the area of a triangle or a quadrilateral, as the case may be.)
2. For each positive integer n , define $a_n = 20 + n^2$, and $d_n = \gcd(a_n, a_{n+1})$. Find the set of all values that are taken by d_n and show by examples that each of these values are attained.
3. Solve for real x :

$$\frac{1}{[x]} + \frac{1}{[2x]} = 9x + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x and $(x) = x - [x]$, [e.g. $[3.4] = 3$ and $(3.4) = 0.4$].

4. In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other.

Show that

- (a) $AD \cdot BC \geq AB \cdot CD$;
- (b) $AD + BC \geq AB + CD$.

5. Let x , y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left| \frac{x}{y} - \frac{y}{x} \right|, \left| \frac{y}{z} - \frac{z}{y} \right|, \left| \frac{z}{x} - \frac{x}{z} \right|$$

can be the lengths of the sides of a triangle.

6. Find the number of unordered pairs $\{A, B\}$ (i.e., the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n -element set X which satisfy the conditions:

- (a) $A \neq B$;
- (b) $A \cup B = X$

[e.g., if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}$, $\{\{a\}, \{b, c, d\}\}$, $\{\phi, \{a, b, c, d\}\}$ are some of the admissible pairs.]

RMO–1998

1. Let $ABCD$ be a convex quadrilateral in which $\angle BAC = 50^\circ$, $\angle CAD = 60^\circ$, $\angle CBD = 30^\circ$, and $\angle BDC = 25^\circ$. If E is the point of intersection of AC and BD , find $\angle AEB$.
2. Let n be a positive integer and $p_1, p_2, \dots, p_n$ be n prime numbers all larger than 5 such that 6 divides $p_1^2 + p_2^2 + \dots + p_n^2$. Prove that 6 divides n .
3. Prove the following inequality for every natural number n :

$$\frac{1}{n+1} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \right) > \frac{1}{n} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n} \right).$$

4. Let ABC be a triangle with $AB = BC$ and $\angle BAC = 30^\circ$. Let A' be the reflection of A in the line BC ; B' be the reflection of B in the line CA ; C' be the reflection of C in the line AB . Show that A', B', C' form the vertices of an equilateral triangle.
5. Find the minimum possible least common multiple (lcm) of twenty (not necessarily distinct) natural numbers whose sum is 801.
6. Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T .

RMO–1999

1. Prove that the inradius of a right-angled triangle with integer sides is an integer.
2. Find the number of positive integers which divide 10^{999} but not 10^{998} .
3. Let $ABCD$ be a square and M, N points on sides AB, BC , respectively, such that $\angle MDN = 45^\circ$. If R is the midpoint of MN show that $RP = RQ$ where P, Q are the points of intersection of AC with the lines MD, ND .
4. If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.
5. If a, b, c are the sides of a triangle prove the following inequality:

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

6. Find all solutions in integers m, n of the equation

$$(m-n)^2 = \frac{4mn}{m+n-1}.$$

7. Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:
 - (a) a, b, c are distinct;
 - (b) $a, b, c \in \{1, 2, 3, \dots, 1999\}$ and
 - (c) $x+1$ divides $ax^2 + bx + c$.

RMO–2000

1. Let AC be a line segment in the plane and B a point between A and C . Construct isosceles triangles PAB and QBC on one side of the segment AC such that $\angle APB = \angle BQC = 120^\circ$ and an isosceles triangle RAC on the other side of AC such that $\angle ARC = 120^\circ$. Show that PQR is an equilateral triangle.
2. Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y .
3. Suppose $(x_1, x_2, \dots, x_n, \dots)$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \dots$, and for all n

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \dots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all k the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \dots + \frac{x_k}{k} \leq 3.$$

4. All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.
5. The internal bisector of angle A in a triangle ABC with $AC > AB$, meets the circumcircle Γ of the triangle in D . Join D to the centre O of the circle Γ and suppose DO meets AC in E , possibly when extended. Given that BE is perpendicular to AD , show that AO is parallel to BD .
6. (i) Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab ?
(ii) Consider two positive integers a and b which are such that $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab ?
7. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

RMO–2001

1. Let BE and CF be the altitudes of an acute triangle ABC , with E on AC and F on AB . Let O be the point of intersection of BE and CF . Take any line KL through O with K on AB and L on AC . Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF . Prove that FM is parallel to EN .
2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.
3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99} \right] = \left[\frac{x}{101} \right]$$

(Here $[z]$ denotes, for any real z , the largest integer not exceeding z ; e. g. $[\frac{7}{4}] = 1$)

4. Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the n numbers $1, 2, \dots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are $1, 2, \dots, n$ in some order.

5. In a triangle ABC , D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72^\circ$.
6. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz.$$

7. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 odd integers by a multiple of 2001.

RMO–2002

1. In an acute triangle ABC , points D, E, F are located on the sides BC, CA, AB respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ABC .

2. Solve the following equation for real x :

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

3. Let a, b, c be positive integers such that a divides b^2 , b divides c^2 and c divides a^2 . Prove that abc divides $(a + b + c)^7$.
4. Suppose the integers $1, 2, 3, \dots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that

$$a_1 < a_2 < a_3 < a_4 < a_5 \text{ and } b_1 > b_2 > b_3 > b_4 > b_5.$$

- (i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \leq j \leq 5$, is at least 6.
- (ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.
5. The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.
6. For any natural number $n > 1$, prove the inequality :

$$\frac{1}{2} < \frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \dots + \frac{n}{n^2 + n} < \frac{1}{2} + \frac{1}{2n}.$$

7. Find all integers a, b, c, d satisfying the following relations :

- (i) $1 \leq a \leq b \leq c \leq d$;
- (ii) $ab + cd = a + b + c + d + 3$.

RMO–2003

1. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90^\circ$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45^\circ$.
2. If n is an integer greater than 7, prove that $\binom{n}{7} - \left\lfloor \frac{n}{7} \right\rfloor$ is divisible by 7. [Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also for any real number x , $[x]$ denotes the greatest integer not exceeding x .]
3. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab, b - bc, c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.
4. Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:
(i) $x \leq y \leq z$;
(ii) $x + y + z \leq 100$.
5. Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \frac{d(B, CA)}{d(P, CA)}, \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. [Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the line YZ .]

6. Find all real numbers a for which the equation

$$x^2 + (a - 2)x + 1 = 3|x|$$

has exactly three distinct real solutions in x .

7. Consider the set $X = \{1, 2, 3, \dots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that
 - (a) $A \cup B = X$;
 - (b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C , $\text{prod}(C)$ denotes the product of all numbers in C ;
 - (c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.

RMO–2004

1. Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ . Prove that there is a point Q on the perpendicular drawn from O to the line l , such that for any point P on the line l , PQ represents the length of the tangent from P to the circle Γ .
2. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T .
3. Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,
 - (a) λ_n is an integer; and
 - (b) $\gcd(\lambda_n, \lambda_{n+1}) = 1$.
4. Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \dots, n\}$ such that $A \cap B \cap C = \phi$, $A \cap B \neq \phi$, $B \cap C \neq \phi$ is $7^n - 2 \cdot 6^n + 5^n$.
5. Let $ABCD$ be a quadrilateral ; X and Y be the midpoints of AC and BD respectively ; and the lines through X and Y respectively parallel to BD, AC meet in O . Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Prove that
 - (a) quadrilaterals $APOS$ and $APXS$ have the same area ;
 - (b) the areas of the quadrilaterals $APOS, BQOP, CROQ, DSOR$ are all equal .
6. Let $(p_1, p_2, p_3, \dots, p_n, \dots)$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1 p_2 \dots p_n + 1$. (Thus $p_2 = 3, p_3 = 7$). Prove that $p_n \neq 5$ for any n .
7. Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that
 - (a) $y < x < 1$; and
 - (b) $x^2 + y^2 < 1$.

RMO–2005

1. Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.
2. If x, y are integers and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.
3. If a, b, c are three real numbers such that $|a - b| \geq c$, $|b - c| \geq a$, $|c - a| \geq b$, then prove that one of a, b, c is the sum of the other two.
4. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)
5. In triangle ABC , let D be the midpoint of BC . If $\angle ADB = 45^\circ$ and $\angle ACD = 30^\circ$, determine $\angle BAD$.
6. Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

7. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

RMO–2006

1. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB . Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are *collinear*.
2. Find the least possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$.
3. If a, b, c are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

4. A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always **two** congruent rectangles.
5. Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD ; $AB = 3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.
6. Prove that there are infinitely many positive integers n such that $n(n + 1)$ can be expressed as a sum of two positive squares in *at least* two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)
7. Let X be the set of all positive integers greater than or equal to 8 and let $f : X \rightarrow X$ be a function such that $f(x + y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

Solutions to CRMO-2007 Problems

1. Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC ; and BE be the altitude from B on AC . Show that $\angle CED > 45^\circ$.
2. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c - a, c - b) = 1$. Suppose there exists an integer d such that $a + d, b + d, c + d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$.
3. Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.
4. How many 6-digit numbers are there such that:
 - (a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$;
 - (b) any digit that appears in the number appears at least twice?(Example: 225252 is an admissible number, while 222133 is not.)
5. A trapezium $ABCD$, in which AB is parallel to CD , is inscribed in a circle with centre O . Suppose the diagonals AC and BD of the trapezium intersect at M , and $OM = 2$.
 - (a) If $\angle AMB$ is 60° , determine, with proof, the difference between the lengths of the parallel sides.
 - (b) If $\angle AMD$ is 60° , find the difference between the lengths of the parallel sides.
6. Prove that:
 - (a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;
 - (b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;
 - (c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solutions to CRMO-2008 Problems

1. Let ABC be an acute-angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N . Prove that ND is equal to the circum-radius of ABC . [15]
2. Prove that there exist two infinite sequences $\langle a_n \rangle_{n \geq 1}$ and $\langle b_n \rangle_{n \geq 1}$ of positive integers such that the following conditions hold simultaneously:
 - (i) $1 < a_1 < a_2 < a_3 < \dots$;
 - (ii) $a_n < b_n < a_n^2$, for all $n \geq 1$;
 - (iii) $a_n - 1$ divides $b_n - 1$, for all $n \geq 1$;
 - (iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \geq 1$.
3. Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that:
 - (i) $0 < 3ab \leq 1$ and
 - (ii) $b \geq \sqrt{3}$.
4. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0,1,2,3 occurs at least once in them. [14]
5. Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c . [17]
6. Find the number of all integer-sided *isosceles obtuse-angled* triangles with perimeter 2008. [16]

Regional Mathematical Olympiad-2009

Problems and Solutions

1. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. Find $\angle BAC$.
2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.
3. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182} .
4. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.
5. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.
 - (i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.
 - (ii) If X and Y are two distinct points inside Γ , prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \leq 1$.
6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

Regional Mathematical Olympiad-2010

Problems and Solutions

1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD , BE , CF are concurrent at O . Suppose the area of triangle OAF is the geometric mean of those of OAB and OEF ; and the area of triangle OBC is the geometric mean of those of OAB and OCD . Prove that the area of triangle OED is the geometric mean of those of OCD and OEF .
2. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.
3. Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.
4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.
5. Let ABC be a triangle in which $\angle A = 60^\circ$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB . Let M be the reflection of A in the line EF . Prove that M lies on BC .
6. For each integer $n \geq 1$, define $a_n = \left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x , for any real number x . Find the number of all n in the set $\{1, 2, 3, \dots, 2010\}$ for which $a_n > a_{n+1}$.

Problems and Solutions: CRMO-2011

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K . Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.
2. Let $(a_1, a_2, a_3, \dots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers $1, 2, 3, \dots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.
3. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n . Prove that $n - kl$ is a perfect square.
4. Consider a 20-sided convex polygon K , with vertices $A_1, A_2, \dots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)
5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B, \angle C$ with B_1 on AC and C_1 on AB . Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB . Prove that $AD = EF$.
6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Problems and Solutions: CRMO-2012, Paper 1

1. Let ABC be a triangle and D be a point on the segment BC such that $DC = 2BD$. Let E be the mid-point of AC . Let AD and BE intersect in P . Determine the ratios BP/PE and AP/PD .
2. Let a, b, c be positive integers such that a divides b^3 , b divides c^3 and c divides a^3 . Prove that abc divides $(a + b + c)^{13}$.
3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \dots, 10\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$.
5. Let ABC be a triangle. Let BE and CF be internal angle bisectors of $\angle B$ and $\angle C$ respectively with E on AC and F on AB . Suppose X is a point on the segment CF such that $AX \perp CF$; and Y is a point on the segment BE such that $AY \perp BE$. Prove that $XY = (b + c - a)/2$ where $BC = a$, $CA = b$ and $AB = c$.
6. Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Problems and Solutions: CRMO-2012, Paper 2

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arc AC internally, the arc BD internally and also touching the side AB . Find the radius of the circle Γ .
2. Let a, b, c be positive integers such that a divides b^4 , b divides c^4 and c divides a^4 . Prove that abc divides $(a + b + c)^{21}$.

3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \dots, 12\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7, 8\}$.
5. Let ABC be a triangle. Let D, E be a points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC . Let BF intersect AD in P and AE in Q respectively. Determine BP/PQ .
6. Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Problems and Solutions: CRMO-2012, Paper 3

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arcs AC and BD both externally and also touching the side CD . Find the radius of the circle Γ .
2. Let a, b, c be positive integers such that a divides b^5 , b divides c^5 and c divides a^5 . Prove that abc divides $(a + b + c)^{31}$.
3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \dots, 10\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{5, 7, 8\}$.
5. Let ABC be a triangle. Let D, E be a points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC . Let BF intersect AD in P and AE in Q respectively. Determine the ratio of the area of the triangle APQ to that of the quadrilateral $PDEQ$.
6. Find all positive integers n such that $3^{2n} + 3n^2 + 7$ is a perfect square.

Problems and Solutions: CRMO-2012, Paper 4

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arc AC externally, the arc BD internally and also touching the side AD . Find the radius of the circle Γ .
2. Let a, b, c be positive integers such that a divides b^2 , b divides c^2 and c divides a^2 . Prove that abc divides $(a + b + c)^7$.
3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \dots, 11\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.
5. Let ABC be a triangle. Let E be a point on the segment BC such that $BE = 2EC$. Let F be the mid-point of AC . Let BF intersect AE in Q . Determine BQ/QF .
6. Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1.$$

Regional Mathematical Olympiad-2017

Time: 3 hours

October 08, 2017

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let AOB be a given angle less than 180° and let P be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB , and $CP : PD = 1 : 2$.

2. Show that the equation

$$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$$

has no solutions in integers a, b .

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x . Find all the real roots of $P(Q(x)) = 0$.
4. Consider n^2 unit squares in the xy -plane centred at point (i, j) with integer coordinates, $1 \leq i \leq n, 1 \leq j \leq n$. It is required to colour each unit square in such a way that when ever $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the three squares with centres at $(i, k), (j, k), (j, l)$ have distinct colours. What is the least possible number of colours needed?
5. Let Ω be a circle with a chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D . Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F . Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω .

6. Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}.$$

आर. एम. ओ. 2017

समय: 3 घंटे

08 अक्टूबर, 2017

निर्देश:

- कैलकुलेटर (किसी भी स्वरूप में) या चांदा लाने की अनुमति नहीं है।
 - रूलर एवं प्रकार लाने की अनुमति है।
 - सभी प्रश्नों के जवाब दें।
 - सभी प्रश्न बराबर अंकों के हैं। अधिकतम अंक: 102
 - हर प्रश्न का हल नए पन्ने से शुरू करें। प्रश्न संख्या का साफ़-साफ़ उल्लेख करें।
- मान लीजिए कि AOB एक कोण दिया हुआ है जो 180° से कम है और P , $\angle AOB$ द्वारा निर्धारित कोणीय क्षेत्र में, एक भीतरी बिंदु है। प्रमाण के साथ दिखाइए कि, केवल रूलर एवं प्रकार का प्रयोग करते हुए, P से गुजरते हुए एक ऐसे रेखाखंड CD की रचना कैसे करेंगे ताकि C अर्ध-रेखा OA पर स्थित हो, D अर्ध-रेखा OB पर स्थित हो, और $CP : PD = 1 : 2$ हो।

- दिखाइए की समीकरण

$$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$$

का पूर्णांकों a, b में कोई हल नहीं है।

- मान लीजिए कि $P(x) = x^2 + \frac{1}{2}x + b$ एवं $Q(x) = x^2 + cx + d$ दो बहुपद हैं जिनके गुणांक वास्तविक हैं और सभी वास्तविक x के लिए $P(x)Q(x) = Q(P(x))$ । इस स्थिति में समीकरण $P(Q(x)) = 0$ के सभी वास्तविक हल ज्ञात कीजिए।
- मान लीजिए कि कार्तीय तल में n^2 इकाई-वर्ग (प्रत्येक का क्षेत्रफल 1 है) दिए गए हैं जिनके केंद्र (i, j) हैं, जहाँ $1 \leq i \leq n, 1 \leq j \leq n$ । प्रत्येक वर्ग को इस प्रकार से रंग से भरना है कि जब भी $1 \leq i < j \leq n$ व $1 \leq k < l \leq n$ हो, तो उन तीनों वर्गों के रंग भिन्न हों जिनके केंद्र $(i, k), (j, k), (j, l)$ हैं। ऐसी स्थिति में न्यूनतम आवश्यक रंगों की संख्या ज्ञात कीजिए।
- मान लीजिए की Ω एक वृत्त है और AB एक जीवा है जो कि व्यास नहीं है। मान लीजिए कि Γ_1 रेखा AB की एक तरफ़ एक वृत्त है जो रेखा AB को C पर स्पर्श करता है और वृत्त Ω को D पर भीतर से स्पर्श करता है। उसी तरह मान लीजिए कि Γ_2 रेखा AB की दूसरी तरफ़ एक वृत्त है जो रेखा AB को E पर स्पर्श करता है और वृत्त Ω को F पर भीतर से स्पर्श करता है। माना लीजिए कि रेखा DC वृत्त Ω से $X \neq D$ पर मिलती है और रेखा FE वृत्त Ω से $Y \neq F$ पर मिलती है। प्रमाणित करिए कि XY वृत्त Ω का व्यास है।
- मान लीजिए कि प्रत्येक वास्तविक संख्या x, y, z संख्या 1 से बड़ी है। प्रमाणित करिए कि:

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

Regional Mathematical Olympiad-2018

1. Let ABC be a triangle with integer sides in which $AB < AC$. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D . Suppose AD is also an integer. Prove that $\gcd(AB, AC) > 1$.
2. Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^n \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

3. For a rational number r , its *period* is the length of the smallest repeating block in its decimal expansion. For example, the number $r = 0.123123123\cdots$ has period 3. If S denotes the set of all rational numbers r of the form $r = 0.\overline{abcdefgh}$ having period 8, find the sum of all the elements of S .
4. Let E denote the set of 25 points (m, n) in the xy -plane, where m, n are natural numbers, $1 \leq m \leq 5$, $1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a, b) , $(a+k, b)$, $(a+k, b+k)$, $(a, b+k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)
5. Find all natural numbers n such that $1 + [\sqrt{2n}]$ divides $2n$. (For any real number x , $[x]$ denotes the largest integer not exceeding x .)
6. Let ABC be an acute-angled triangle with $AB < AC$. Let I be the incentre of triangle ABC , and let D, E, F be the points at which its incircle touches the sides BC, CA, AB , respectively. Let BI, CI meet the line EF at Y, X , respectively. Further assume that both X and Y are outside the triangle ABC . Prove that
 - (i) B, C, Y, X are concyclic; and
 - (ii) I is also the incentre of triangle DYX .

Regional Mathematical Olympiad-2019

Time: 3 hours

October 20, 2019

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.
2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC . Extend AG , BG and CG to meet the circle Ω again in A_1 , B_1 and C_1 , respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.
3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{b^2 + c^3 + a^3} + \frac{c}{c^2 + a^3 + b^3} \leq \frac{1}{5abc}.$$

4. Consider the following 3×2 array formed by using the numbers 1, 2, 3, 4, 5, 6:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3 \times k}$ for a suitable k , adding more columns, using the numbers 7, 8, 9, ..., $3k$ such that

$$\sum_{j=1}^k a_{1j} = \sum_{j=1}^k a_{2j} = \sum_{j=1}^k a_{3j} \quad \text{and} \quad \sum_{j=1}^k (a_{1j})^2 = \sum_{j=1}^k (a_{2j})^2 = \sum_{j=1}^k (a_{3j})^2.$$

5. In an acute angled triangle ABC , let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC , respectively. Let X, Y be feet of altitudes from L, N on to the line DF . Prove that XM is perpendicular to MY .
6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.